Compact Spaces

In analysis, we learn that $[a,b]$ is compact, and, more generally any closed bounded subset of $\mathbb{R}^n$ is compact.

Good properties of compact spaces:

- generalize the metric space notion of "boundedness"
- Any continuous map $f: \underset{\substack{\uparrow \\ \text{compact}}}{K} \to \mathbb{R}$ achieves its maximum...

Def: X a topological space. A collection of open sets $\{U_i\}_{i \in J}$ is an *open cover* if $\bigcup_{i \in J} U_i = X$.

Def: X is *compact* if every open cover contains a finite subcollection that also covers X.

i.e. $\exists \underset{\substack{\uparrow \\ \text{subcollection} \\ \text{of open sets}}}{\mathcal{A}} \subseteq \{U_i\}_{i \in J}$ finite s.t. $\bigcup \mathcal{A} = X$.

Ex: $\mathbb{R}$ is not compact. The covering $\mathbb{R} = \bigcup_{\mathbb{Z}} (n, n+2)$ has no finite subcover.

$(0,1]$ is not compact... $\bigcup (1/n, 1]$ has no finite subcover.

Ex: $X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ is compact!

If $X = \bigcup U_i$ covers X, there is some U_j containing 0.

U_j contains all but finitely many of the remaining points, so choose an open neighborhood of each of the remaining points. Those and U_j form a finite subcover.

Some basic results about compact spaces:

Thm: If A is compact and $f: A \longrightarrow X$ continuous, then $f(A)$ is compact.

Pf: Let $\bigcup U_i$ be an open cover of $f(A)$. Then $\bigcup f^{-1}(U_i)$ is an open cover of A, which has a finite subcover
$$\bigcup_{j \in J \leftarrow \text{finite}} f^{-1}(U_j).$$

$U_j = f(f^{-1}(U_j))$, so the sets U_j, $j \in J$ cover $f(A)$. □

Thm: $[0,1]$ is compact.

Pf: Let $\{U_i\}$ be an open cover of $[0,1]$.

Let $A = \{x \in [0,1] \mid \exists \text{ a finite subcover of } [0,x]\}$

Clearly $0 \in A$, so $A \neq \phi$.

We'll show A is open and closed. Since $[0,1]$ is connected, this will imply $A = [0,1]$.

If a cover works for $[0,x]$, then $x \in U_i$ for some i, so $B_r(x) \subseteq U_i$, some $r > 0$.

So $x \in B_{r/2}(x) \subseteq A$, so A is open.

To show A is closed, suppose x is a limit point of A. $x \in U_i$ some i, so $x \in B_r(x) \subseteq U_i$, some r.

$[0, x - r/2]$ admits a finite cover, so if we add U_i to the finite cover, we get a finite cover of $[0, x]$. □

You may recall from analysis that $A \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ A is closed and bounded.

In general, how do closed sets relate to compact sets?

Thm: If X is compact, then any closed $A \subseteq X$ is compact.

Pf: Let $\mathcal{A}$ be an open covering of A by sets open in X. Then $\mathcal{A} \cup \{X - A\}$ is an open covering of X.

Let $\mathcal{B}$ be a finite subcovering. Then $\mathcal{B}$ covers A (after possibly removing $X - A$), so $\mathcal{B}$ is a finite subcover. □

Are all compact sets closed? Sadly, not in general:

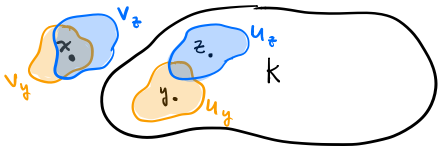
Ex: $X \subseteq \mathbb{R}$ w/ the cofinite topology is always compact (by a HW problem) but not always closed.

In Hausdorff spaces, it is true though!

Thm: If X is a Hausdorff space, then any compact set $K \subseteq X$ is closed.

Pf: We'll show $X \setminus K$ is open.

Let $x \in X \setminus K$. For every $y \in K$ there are disjoint neighborhoods $U_y \ni y$ and $V_y \ni x$. K is covered by $\bigcup U_y$.



Thus, K has a finite subcover, say $U_1 \cup U_2 \cup \dots \cup U_k$, w/ corresponding neighborhoods $V_1, V_2, \dots, V_k$ of x w/ U_i disjoint from V_i. Then $V = V_1 \cap \dots \cap V_k$ is a finite intersection of open sets, and is thus an open neighborhood of x.

Moreover, V is disjoint from $U_1 \cup \dots \cup U_k$, so $x \in V \subseteq X \setminus K$. Thus $X \setminus K$ is open, so K is closed. $\square$

It's much easier to check if maps between compact Hausdorff spaces are homeomorphisms:

Thm: A continuous bijection $f : X \to Y$ between compact Hausdorff spaces is a homeomorphism.

Pf: We want to show that the images of open sets are open, so it suffices to show the images of closed sets are closed.

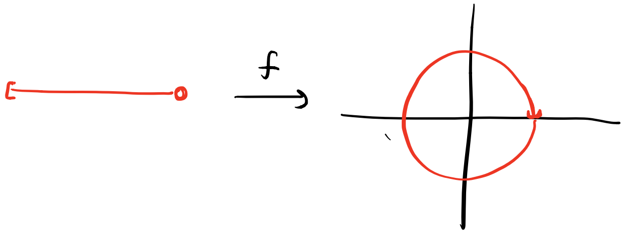
Let $A \subseteq X$ be closed. Then A is compact. Thus $f(A)$ is compact $\Rightarrow$ $f(A)$ is closed, since Y is Hausdorff. $\square$

Note: This only works for compact Hausdorff spaces.

Consider the bijection

$$f : \overset{\substack{\text{Hausdorff,}\\ \text{not compact}\\ \downarrow}}{[0,1)} \longrightarrow \overset{\substack{\text{compact}\\ \text{Hausdorff}\\ \downarrow}}{S^1}, \quad \text{defined} \quad x \longmapsto (\cos(2\pi x), \sin(2\pi x))$$

This is continuous but certainly not a homeomorphism



Products of compact spaces

Thm: If X and Y are compact, then $X \times Y$ is compact.

Pf: Let $\mathcal{A}$ be an open cover of $X \times Y$. Each element of $\mathcal{A}$ is the union of basis elements of the form $U \times V$, where

$$U \subseteq X \text{ and } V \subseteq Y \text{ are open.}$$

If those basis elements have a finite subcover, then $\mathcal{A}$ does as well. (by replacing $U \times V$ w/ the open set in $\mathcal{A}$ in which it is contained).

Thus, we can assume $\mathcal{A}$ consists of basis elements (of form $U_i \times V_i$)

For $x \in X$, $\{x\} \times Y$ is homeomorphic to Y (see HW), so $\{x\} \times Y$ is compact. so it has a finite subcover of the form $\bigcup_{i=1}^{n} U_i \times V_i$, where $x \in U_i$ $\forall i$.

Then $W_x = \bigcap U_i$ is a neighborhood of x, and $\bigcup_{i=1}^{n} U_i \times V_i$ is a finite cover of $W_x \times Y$.

Thus, for each $x \in X$, there's a finite subcover of $W_x \times Y$. But the W_x cover X, so finitely many of them cover X.

Thus, $\exists$ $x_1, \dots, x_m \in X$ s.t. $W_{x_1} \cup \dots \cup W_{x_m} = X$ and there is a finite subcovering of $\mathcal{A}$ that covers $W_{x_i} \times Y$. The union of these is a finite subcovering of $X \times Y$. □

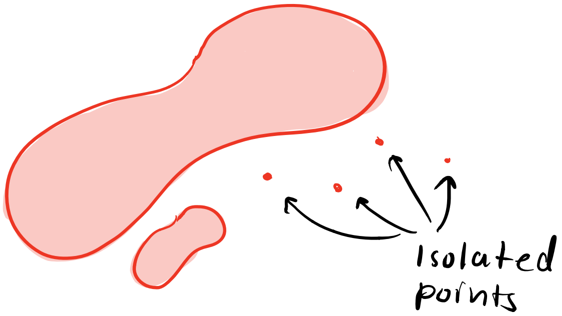
Cor: The product of finitely many compact spaces is compact.

In fact the product of infinitely many compact spaces (given the product topology) is compact! This is a deep result called Tychonoff's Theorem. Proof requires Axiom of choice. (see section 37 of Munkres for proof.)

Uncountability of $\mathbb{R}$

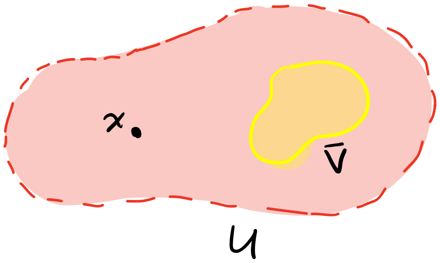
We can use properties of compact Hausdorff spaces to give a slick proof that $\mathbb{R}$ is uncountable.

Def: X a topological space. $x \in X$ is an <u>isolated point</u> if $\{x\}$ is open in X.



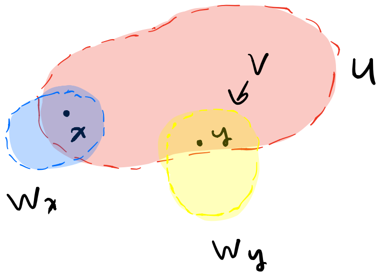
Theorem: If X is a nonempty compact Hausdorff space with no isolated points, then X is uncountable.

Pf: Claim: If $U \subseteq X$ open, $x \in X$, $\exists$ V open s.t. $\emptyset \neq V \subseteq U$ and $x \notin \overline{V}$.



Pf of Claim: Choose $y \in U$ s.t. $x \neq y$. This is possible since U can't be a one-point set.

We can find $W_x \ni x$ and $W_y \ni y$ disjoint neighborhoods. Then set $V = W_y \cap U$. $x \notin \overline{V}$, since W_x is open and disjoint from V.



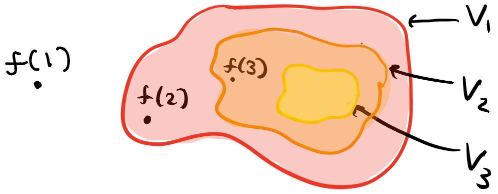
Now we use the claim to show uncountability:

Let $f: \mathbb{Z}_+ \to X$ be some function.

By the claim, set $U = X$, and find $V_1 \subseteq X$ s.t. $\overline{V}_1$ doesn't contain $f(1)$.

For $n > 1$, apply the claim to $f(n)$ and $U = V_{n-1}$.

Then $\overline{V}_1 \supset \overline{V}_2 \supset \dots$ are nonempty closed sets w/ $f(n) \notin \overline{V}_n$.



If $\bigcup (X \setminus \overline{V}_i) = X$, any finite subcover would leave out some nonempty $\overline{V}_i$. Thus, $\bigcup (X \setminus \overline{V}_i) \neq X$, so $\bigcap \overline{V}_i \neq \emptyset$.

Take $x \in \bigcap \overline{V}_i$. $x \neq f(n)$ for any n, so $f: \mathbb{Z}_+ \to X$ is <u>not</u> surjective, so X is uncountable. $\square$

Cor: Every closed interval in $\mathbb{R}$ is uncountable.